Note on Numerical Integration: Applications

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1 Introduction

We see how various numerical integration methods are used macroeconomic models.

2 iid Normal Shock

2.1 Motivating Example

Let's consider the following dynamic optimization problem:

Problem 1 (Consumption-Savings Decision with iid Normal Shock to Labor Income)

$$V(z,a) = \max_{c,a'} \{u(c) + \beta E_z V(z',a')\}$$

subject to

$$a(1+r) + we^z = a' + c$$
$$c \geq 0$$
$$a' \geq 0$$
$$z \sim iidN(0,\sigma_z^2)$$
r and w are given

Assume standard functional form of CRRA with risk aversion parameter σ. This is standard savings-consumption choice problem with iid normal shock to (log of) labor income. Apart from the expectation operator, we can easily solve the problem, using dynamic optimization method.

2.2 Algorithm

How to numerically compute the expected value? First of all, notice that what the expectation operator does is to integrate. So we can rewrite the expectation term:

$$E_z V(z',a') = \int_{z'} V(z',a') f(z'|z) dz'$$

Let's see how to compute the integral numerically. There are a couple of ways to do it. We see *(i)* simple discretization, *(ii)* Gauss-Hermite quadrature method (Details can be found in the note on Gaussian quadrature), and *(iii)* method based on Monte-Carlo simulation.

Algorithm 1 (Simple Discretization for iid Normal Shock)

1. *Put the upperbound $\overline{z}$ and the lowerbound $\underline{z}$ to the distribution of z. Considering the symmetry of the normal distribution, a natural way to fix the bounds is to fix λ such that:*

$$\overline{z} = \lambda\sigma_z \qquad \underline{z} = -\lambda\sigma_z$$

2. *Set m, which is the number of potential realizations of z.*

3. *Set $\{z_i\}_{i=1}^m$ such that, $z_1 = \underline{z}$, $z_m = \overline{z}$, and all of $\{z_i\}_{i=1}^m$ are equally distanced. In other words, for $i = 1, 2, ..., m$:*

$$z_i = \underline{z} + \frac{\overline{z} - \underline{z}}{m-1}(i-1) = \underline{z} + \frac{2\lambda\sigma_z}{m-1}(i-1)$$

4. *Denote the probability of z_i as p_i. In addition, denote the cumulative density function (CDF) of the normal distribution with mean μ and variance σ^2 $F(., \mu, \sigma^2)$, then (in our example $\mu = 0$, but we write it in a general way). For $i = 2, 3, ..., m-1$, p_i can be computed as follows:*

$$p_i = F(\frac{z_i + z_{i+1}}{2}, 0, \sigma_z^2) - F(\frac{z_{i-1} + z_i}{2}, 0, \sigma_z^2)$$

In other words, the probability that z_i is realized is defined as the probability that a draw from the normal distribution falls into the interval constructed around z_i. For the two endpoints, we have to take care of the fact that we cut both tails by putting finite bounds to the space of realizations. Therefore:

$$p_1 = F(\frac{z_1 + z_2}{2}, 0, \sigma_z^2)$$

$$p_m = 1 - F(\frac{z_{m-1} + z_m}{2}, 0, \sigma_z^2)$$

Notice that there is a freedom in how to choose abscissas in the method above. What we learned from Gaussian quadrature method is that we can improve the accuracy of numerical integration formula by optimally choosing the location of abscissas. This is the spirit of the next method, which uses Gauss-Hermite quadrature.

Algorithm 2 (Gauss-Hermite Quadrature for iid Normal Shock)

1. *Choose N. For reducing the computational cost, it's better to have smaller N. But there is a usual trade-off between cost and precision. Unless you are severely constrained by computational cost (there is a severely binding upperbound for N), try some number, like 5, and keeps increasing until the answer becomes insensitive to an increase of N.*

2. *Compute N roots of order N Hermite polynomial $P_N(x)$. Denote them as $\{x_i\}_{i=1}^N$. Since there is no simple formula for the roots, the easiest way is to find in a book on numerical methods.*

3. *Set $\{\omega_i\}_{i=1}^N$ as follows:*

$$\omega_i = \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{x - x_j}{x_i - x_j} \; e^{-x^2} \; dx$$

Again, the simplest way to obtain $\{\omega_i\}_{i=1}^N$ is to find in a book. Since the same values of ω_i are used for all Gauss-Hermite quadrature, it's easy to find in a book on numerical methods.

4. *Construct* $\{z_i\}_{i=1}^N$ *using:*

$$z_i = \sqrt{2}\sigma x_i + \mu$$

5. *The expected value, given* a'*, can be computed by:*

$$E_z V(z', a') = \int_{z'} V(z', a') f(z'|z) dz' = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} V(z', a') \exp - \left(\frac{z-\mu}{\sqrt{2}\sigma}\right)^2 dz' \simeq \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \omega_i V(z_i', a')$$

6. *Since the distribution of the shock is independent from other variables in the model, for any* a*, we only need to find* $V(z_i', a')$ *for* $i = 1, 2, .., N$ *to compute the expected value.*

Let's redefine the weights for $i = 1, 2, ..., N$ as follows:

$$\tilde{\omega_i} = \frac{\omega_i}{\sqrt{\pi}}$$

Then the We have the following relationship:

$$E_z V(z', a') \simeq \sum_{i=1}^{N} \tilde{\omega_i} V(z_i', a')$$

This implies that, we can interpret the procedure above as the discretization of the iid shock. The iid normal shock z is discretized and now is characterized by N potential realizations $\{z_i\}_{i=1}^N$ and probability attached to each of the realization $\{\tilde{\omega}_i\}_{i=1}^N$.

As a preparation for using Monte Carlo simulation, let's redefine the problem as follows:

Problem 2 (Consumption-Savings Decision with iid Normal Shock to Labor Income)

$$EV(a) = Emax_{c,a'} \{u(c) + \beta EV(a')\}$$

subject to

$$a(1+r) + we^z = a' + c$$
$$c \geq 0$$
$$a' \geq 0$$
$$z \sim iidN(0, \sigma_z^2)$$
$$r \text{ and } w \text{ are given}$$

The formulation exploits the fact that the future realization of the shocks do not depend on the current realization of the shock, because of the iid assumption. In the formulation above, what we iterate on is the expected value ($EV(a')$), which only depends on a', because z' is integrated away. We will see how Monte-Carlo simulation is applied to this problem:

Algorithm 3 (Monte Carlo Simulation for iid Normal Shock)

1. *Suppose we have a guess for* $EV(a)$*.*

2. *Set* n*, which is the number of draws. We are going to implement step 3-4 for* n *times.*

3. *Draw z_i from $N(0, \sigma^2)$. We need a random number generator associated with a normal distribution. Some computer languages have a built-in function which doing this. For others, it might be the case that only a random number generator from uniform $[0, 1]$ is available. In this case, we need to use a subroutine which converts a draw from uniform $[0, 1]$ into a draw from a normal distribution. There are many subroutines which do this job.*

4. *For a a, given z_i, and given the guess $EV(a')$, solve the optimization problem in the Bellman equation. Call the resulting value as $V(a, z_i)$.*

5. *Updated guess of $EV(a)$ can be computed by:*

$$EV(a) = \frac{1}{n} \sum_{i=1}^{n} V(a, z_i)$$

6. *Notice that the frequency of each realization is taken care of by drawing z_i form the exact distribution of z.*

Notice that the numerical integration formula is a random variable in Monte-Carlo method in general. For the current case, it might be hard to see the benefit of using Monte Carlo simulation. However, the method becomes more effective if we deal with multiple shocks. Suppose we have m iid shocks in the optimization problem. If we discretize each of the shock using n abscissas, we need to evaluate n^m values associated with different realizations of the shocks. However, as we increase the number of shocks, importance of each point declines, because the probability that each of the realizations occurs decline. The great benefit of Monte Carlo simulation is that the procedure of drawing shocks from the exact distribution takes care of the decline of importance of each realization.

In general, in order to keep the same order of approximation error using quadrature method, we need to keep using the same number of abscissas for each dimension. It means that, to keep the order of the approximation error, the number of abscissas should increase exponentially (n^m). On the other hand, in Monte Carlo simulation, the magnitude of the approximation error is independent of the number of states. In other words, as long as the number of draws are kept the same, we can get the same order of approximation error regardless of the number of shocks.

That's exactly why Monte-Carlo simulation is said to be able to break "the curse of dimensionality". Monte-Carlo simulation method is very efficient if there are large number of shocks in the model. More discussion can be found in Rust (ECO1997).

3 Markov Chain

Next type of shock that we want to deal with is Markov process. We start by reviewing properties of Markov process in general and also those of Markov chain, which is what we get when we discretize a general Markov process.

3.1 Markov Process

An important element of the recursive method is Markov process. Markov process is a stochastic process with the Markov property. A stochastic process has the Markov property if the probability

distribution of the realization in the next period (z_{t+1}) only depends on the realization in the current period (z_t). This feature is crucial because then the shock has a recursive formula, where only the most recent realization matters.

Actually, Markov property is less restrictive than it seems. Even if the probability distribution depends of z_{t+1} depends on both z_t and z_{t-1}, we can define a Markov process by redefining $\tilde{z}_t = (z_{t-1}, z_t)$.

The most popular Markov process in macro literature is AR(1) process. AR(1) is one of the simplest form with persistence and Markov property. AR(1) takes the following form:

$$z_{t+1} = \rho z_t + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma^2)$$

where ρ is a parameter characterizing the persistence of the shock. It's easy to see the following properties:

1. Unconditional mean of z_t is zero.
2. Mean of z_{t+1} conditional on z_t is ρz_t.
3. Unconditional variance of z_t is $\frac{\sigma^2}{1-\rho^2}$.
4. Variance of z_{t+1} conditional on z_t is σ^2.
5. Autocorrelation of z_t is $\frac{\rho\sigma^2}{1-\rho^2}$.

3.2 Markov Chain

Markov chains are discrete-valued Markov processes. When we compute expectation of value functions using numerical integration methods, what we are doing usually is to convert continuous values process (like AR(1)) into discretized version of the process and use it to compute expected values. The followings are the properties of Markov chains:

1. The realization of z is restricted to a finite set $\{z_1, z_2, ..., z_n\}$.
2. The dynamics of the process is characterized by a so-called transition matrix $P = (p_{i,j})$, where each element $p_{i,j}$ represents the probability $z' = z_j$ conditional on $z = z_i$.
3. Naturally, for all $i = 1, 2, ..., n$:

$$\sum_{j=1}^{n} p_{i,j} = 1$$

4. Denote a probability distribution z as $\pi = (\pi_1, \pi_2, ..., \pi_n)$, then the probability distribution of z' (π') can be characterized by:

$$\pi'_j = \sum_{i=1}^{n} p_{i,j}\pi_i \qquad j = 1, 2, ..., n$$

Regarding the stationary distribution of z, we have the following theorems (Ljungqvist and Sargent: Theorem 1 and 2 in Chapter 1):

Theorem 1 (Stationary Distribution of Markov Chain (1))
For some Markov chain with the transition matrix P, if $p_{i,j} > 0$ for all i and j, P has a unique stationary distribution π^∞, and any π^0 converges to π^∞.

Theorem 2 (Stationary Distribution of Markov Chain (2))
For some Markov chain with the transition matrix P, and $n \geq 1$, if $p^n_{i,j} > 0$ for all i and j, P has a unique stationary distribution π^∞, and any π^0 converges to π^∞.

3.3 Computation of Stationary Distribution

There are three ways to compute the stationary distribution of a Markov chain.

1. **Updating distribution:**
   (a) Set an initial distribution π^0
   (b) Update π^0 and obtain π^1, using P
   (c) Compare π^0 and π^1. If they are close enough, stop. Otherwise let $\pi^0 = \pi^1$ and go back to step (b).

2. **Monte-Carlo simulation:**
   (a) Prepare n agents which are characterized by z. Type of agents can be stored by $\{z_i\}_{j=1}^n$
   (b) The distribution can be defined as:

$$\pi^0_i = \frac{\sum_{j=1}^n I_{z_j = z_i}}{n} \qquad \forall i$$

   where $I_{[\text{condition}]}$ is 1 if [condition] is true, and 0 otherwise.
   (c) Update z of each agent. With a random number generator drawn from a uniform distribution over $[0, 1]$, use the following procedure to update z and obtain z':
      i. Suppose $z = z_i$. Set $j = 1$.
      ii. Use a random number generator and obtain $e \in [0, 1]$.
      iii. If $\sum_{k=1}^j p_{i,k} \geq e$, then $z' = z_j$.
      iv. Otherwise update j by $j = j + 1$ and go back to step iii.
   (d) Compute new distribution using the updated states of agents and the following formula:

$$\pi^1_i = \frac{\sum_{j=1}^n I_{z'_j = z_i}}{n} \qquad \forall i$$

   (e) Compare π^0 and π^1. If they are close, done. Otherwise, Update agents' state by replacing z with z' and go back to step (b).

3. **Using Eigenvector:** By the definition of the stationary distribution, the stationary distribution π^* satisfies the following:

   $$\pi^{*\prime} = \pi^{*\prime} P$$

   This is equivalent to:

   $$(I - P')\pi^* = 0$$

   Using this formula, we can find π^* by finding the eigenvector associated with unit eigenvalue of -P', normalized such that the following holds.

   $$\sum_i^n \pi_i^* = 1$$

4 AR(1) Shock

4.1 Motivating Example

Let's consider the following dynamic optimization problem:

Problem 3 (Consumption-Savings Decision with AR(1) Shock to Labor Income)

$$V(z, a) = \max_{c,a'} \{u(c) + \beta E_z V(z', a')\}$$

subject to

$$a(1 + r) + we^z = a' + c$$
$$c \geq 0$$
$$a' \geq -\overline{a}$$
$$z' = \rho z + \epsilon \qquad \epsilon \sim N(0, \sigma^2)$$
r and w are given

This is basically a part of the standard incomplete market general equilibrium model, developed by Aiyagari (1994). In the Aiyagari's model, r and w are determined such that the labor and capital markets are clear. Agents take them as given because they are atomless and does not affect the prices.

First of all, notice again the expected value is just an integral of the value with respect to the shock. Therefore:

$$E_z V(z', a') = \int_{z'} V(z', a') f(z'|z) dz'$$

How can we compute numerically the expected value in the Bellman equation? Similar to the case with iid normal shock, three ways are introduced: Tauchen (1986), Tauchen and Hussey (1991), and Monte Carlo simulation. Let's look at them one by one.

4.2 Tauchen (1986)

The method is motivated by the property of the AR(1) process, that, conditional on the current realization of z, z' follows a normal distribution with mean ρz and variance σ^2. The method uses the function which calculates the cumulative density function of normal distribution, which is not trivial but usually provided in most of the programming languages we use. The algorithm is the following:

Algorithm 4 (Tauchen (1986))

1. *Put the upperbound $\overline{z}$ and the lowerbound $\underline{z}$ to the distribution of z. Taking into account that the unconditional mean and variance of z is 0 and $\sigma_z^2 = \frac{\sigma^2}{1-\rho^2}$, Tauchen (1986) suggests:*

   $$\overline{z} = \lambda\sigma_z \qquad \underline{z} = -\lambda\sigma_z$$

   with $\lambda = 3$.

2. *Set m, which is the number of potential realizations of z. Tauchen (1986)'s suggestion is that $m = 9$ does a good job.*

3. *Set $\{z_i\}_{i=1}^m$ such that, $z_1 = \underline{z}$, $z_m = \overline{z}$, and all of $\{z_i\}_{i=1}^m$ are equally distanced. More precisely:*

   $$z_i = \underline{z} + \frac{\overline{z} - \underline{z}}{m-1}(i-1) = \underline{z} + \frac{2\lambda\sigma_z}{m-1}(i-1)$$

4. *Define $s = \frac{z_2 - z_1}{2}$. This is half of the distance between two adjacent points and should be the same for all $\{z_i\}_{i=1}^m$.*

5. *The AR(1) process is now characterized by the finite set of realizations $\{z_i\}_{i=1}^m$, and the transition probability $P = (p_{i,j})$ where $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$.*

6. *Pick $i \in \{1, 2, 3, ..., m-1, m\}$, and $j \in \{2, 3, ..., m-1\}$. The transition probability from moving from z_i to z_j is defined as the probability that $z' \in [z_j - s, z_j + s]$, conditional on $z = z_i$. If we denote the cumulative density function (CDF) of the normal distribution with mean μ and variance σ^2 $F(., \mu, \sigma^2)$, then:*

   $$p_{i,j} = F(z_j + s, 0, \sigma^2) - F(z_j - s, 0, \sigma^2)$$

7. *For $j = 1$ and $j = m$, we need to take care of the tails, so:*

   $$p_{i,1} = F(z_1 + s, 0, \sigma^2)$$

   $$p_{i,m} = 1 - F(z_m - s, 0, \sigma^2)$$

If we use the method, the space of z is discretized into $\{z_i\}_{i=1}^m$. Therefore, the value function becomes: $V(z_i, a)$. And the expectation is with respect to z_i, instead of z. Once P and $\{z_i\}_{i=1}^m$ are obtained, we can compute the expected value in a way such that the shock follows a Markov chain characterized by $\{z_i\}_{i=1}^m$ and P. More formally:

$$E_{z_i} V(z', a') \simeq \sum_{j=1}^{m} p_{i,j} V(z_j, a')$$

4.3 Tauchen and Hussey (1991)

The second method is again based on the Gauss-Hermite quadrature. The method enables us to choose the abscissas efficiently. It's originally developed in Tauchen and Hussey (ECO1991) and Burnside (Chapter 5 of "Computational Methods of the Study of Dynamic Economics", edited by Marimon and Scott, 1999) describes the method using simpler and more familiar examples.

First of all, remember the iid normal example. If we use Gauss-Hermite quadrature formula, we can obtain the following discretization formula:

$$E_{z_i}V(z',a') \simeq \frac{1}{\sqrt{\pi}}\sum_{j=1}^{n}\omega_j V(z_j,a')$$

where $\{\omega_i\}_{j=1}^n$ is defined in the same way as for the iid case.

However, there is one problem. For AR(1), the conditional mean of z' is different depending on the current z. Therefore, the abscissas that we use to compute numerical integration are different for different current z. This is not convenient for us, because we want the set of the potential realization of z to be stationary. Otherwise, the set of z that we need to update the expected value increases without bound. In order to overcome this problem, Tauchen and Hussey proposes the following integration formula.

$$\begin{aligned}
E_{z_i}V(z',a') &= \int_{z'} V(z',a')f(z'|z)dz' \\
&= \int_{z'} V(z',a')\frac{f(z'|z)}{f(z'|\overline{z})}f(z'|\overline{z})dz'
\end{aligned}$$

If we interpret the original integration problem as integrating $V(z',a')\frac{f(z'|z)}{f(z'|\overline{z})}$ using $f(z'|\overline{z})$ as the weighting function, we always have the same set of abscissas regardless of the current z, because the weighting function used here is $f(z'|\overline{z})$, which is independent of the current z. If we use this formula, the Gauss-Hermite quadrature gives:

$$\begin{aligned}
E_{z_i}V(z',a') &= \int_{z'} V(z',a')f(z'|z)dz' \\
&= \int_{z'} V(z',a')\frac{f(z'|z)}{f(z'|\overline{z})}f(z'|\overline{z})dz' \\
&\simeq \frac{1}{\sqrt{\pi}}\sum_{j=1}^{n}\omega_j V(z_j,a')\frac{f(z_j|z_i)}{f(z_j|\overline{z})} \\
&= \sum_{j=1}^{n}\tilde{\omega}_{i,j}V(z_j,a')
\end{aligned}$$

where

$$z_j = \sqrt{2}\sigma x_j + \overline{z} \quad \forall j$$
$$\tilde{\omega}_{i,j} = \frac{1}{\sqrt{\pi}}\omega_j \frac{f(z_j|z_i)}{f(z_j|\overline{z})} \quad \forall i,j$$
$$\omega_i = \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x - x_j}{x_i - x_j}\, e^{-x^2}\, dx \quad \forall i$$

and $\overline{z} = 0$, and $\{x_i\}_{i=1}^n$ are the roots of order n Hermite polynomial. Finally, we want to be able to compare the $\{\tilde{\omega}_{i,j}\}$ to be comparable to the transition matrix of a Markov chain. To achieve that, we need to normalize $\{\tilde{\omega}_{i,j}\}$. Let's summarize the algorithm:

Algorithm 5 (Tauchen and Hussey (1991))

1. *Suppose the AR(1) process that we want to discretize is the following:*

   $$z' = (1-\rho)\overline{z} + \rho z + \epsilon \quad \epsilon \sim N(0, \sigma^2)$$

   This formula implies that the unconditional mean of z is $\overline{z}$. If $\overline{z} = 0$, we go back to the simpler formula of:

   $$z' = \rho z + \epsilon$$

2. *Choose n.*

3. *Compute (obtain) n roots of order n Hermite polynomial $P_n(x)$. Denote them as $\{x_i\}_{i=1}^n$.*

4. *Set the Gauss-Hermite weight $\{\omega_i\}_{i=1}^n$ as follows:*

   $$\omega_i = \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x - x_j}{x_i - x_j}\, e^{-x^2}\, dx$$

5. *Construct $\{z_i\}_{i=1}^n$ using:*

   $$z_i = \sqrt{2}\sigma x_i + \overline{z}$$

6. *Construct $\{\tilde{\omega}_{i,j}\}$ as follows:*

   $$\tilde{\omega}_{i,j} = \frac{\omega_j}{\sqrt{\pi}} \frac{f(z_j|z_i)}{f(z_j|\overline{z})} \quad \forall i,j$$

7. *Normalize $\{\tilde{\omega}_{i,j}\}$ to construct $\{p_{i,j}\}$. In particular, the following formula is used to construct $\{p_{i,j}\}$:*

   $$p_{i,j} = \frac{\tilde{\omega}_{i,j}}{\sum_{j=1}^n \tilde{\omega}_{i,j}}$$

8. *The expected value, given z_i and a', is given by:*

$$E_{z_i}V(z', a') \simeq \sum_{j=1}^{n} p_{i,j}V(z_j, a')$$

A couple of comments:

1. The method virtually approximate the original AR(1) process by n state Markov chain with transition probability $\{p_{i,j}\}$.
2. Because of the normalization, the approximated integral does not necessarily carry the good properties of the Gauss-Hermite quadrature. But it can be shown that, as n is increased, the approximated integral converges to the integral associated with Gauss-Hermite quadrature.

4.4 Monte-Carlo simulation

Let us skip the detailed explanation as it's a simple extension of the iid normal case.

5 Random Walk Shock

5.1 Motivating Example

Random walk is tricky because it cannot be used nicely with a model with infinitely-lived agents. This is because there is no stationary distribution associated with a random walk (cross-sectional variance goes to infinity). However, it's totally doable to use a random walk process with life-cycle model (a model with finite lifetime).

Random walk shock is particularly important because in the labor literature, random walk process is often used to capture permanent shock to (log of) wage or labor income.

Therefore, our example is a component of a life-cycle model. General equilibrium version of the model is developed by Huggett (1996) and Rios-Rull (1996).

Problem 4 (Consumption-Savings Decision in a Life-Cycle Model)

$$V(i, z, a) = \max_{c,a'} \{u(c) + \beta E_z V(i+1, z', a')\}$$

subject to

$$a(1+r) + we^z = a' + c$$
$$c \geq 0$$
$$a' \geq 0$$
$$z' = z + \epsilon \qquad \epsilon \sim N(0, \sigma^2)$$
$$r \text{ and } w \text{ are given}$$
$$i = 1, 2, .., I$$

There are a couple of ways to compute the expected value in the above problem. We overview them.

5.2 Computation of Expected Value

1. Since z' is normally distributed, conditional on z, we can use the method that we used for normal shock. Suppose we use n abscissas to numerically compute integral. In the next period, the potential realization of z' is different for different z. Therefore, the number of necessary abscissas increases exponentially. Since we have I periods as the lifetime, if we assume that all the agents have the same initial z, the total number of abscissas across all age $(i = 1, 2, ..., I)$ is:

   $$n + n^2 + n^3 + ... + n^I = \frac{n(n^I - 1)}{n - 1}$$

   which is really large.

2. Instead, we can always use equally distanced abscissas. Then, many of the abscissas coincide, which enables us to reduce the total number of abscissas. Let's discretize the potential realizations of innovation term ϵ as $\{-m\sigma, -(m-1)\sigma, ..., 0, \sigma, 2\sigma, ..., m\sigma\}$.

   Suppose, initially, all agents have the same $z = \overline{z}$ The set of possible realizations of z in the next period is:

   $$\{\overline{z} - m\sigma, \overline{z} - (m-1)\sigma, ..., \overline{z}, \overline{z} + \sigma, \overline{z} + 2\sigma, ..., \overline{z} + m\sigma\}$$

   Notice that one of the abscissas is shared with the set of possible realization in the previous period.

   In the next period, it is easy to see that many of the abscissas are shared. At the end, the set of possible realizations, for all possible z is:

   $$\{\overline{z} - 2m\sigma, \overline{z} - (2m-1)\sigma, ..., \overline{z}, \overline{z} + \sigma, \overline{z} + 2\sigma, ..., \overline{z} + 2m\sigma\}$$

   If we keep doing this, the size of potential realizations of z associated with age i is:

   $$1 + 2m(i-1)$$

   Therefore, with $i = I$, we have $1 + 2m(I-1)$ abscissas to be used.

   If we sum up the number of abscissas for all ages, we get:

   $$1 + (1 + 2m) + (1 + 4m) + (1 + 6m) + ... + (1 + 2m(I-1)) = I + m(I-1)$$

   which is substantially smaller than the number for the previous method.

   For obtaining the transition matrix, we can use the simple discretization method that we used for iid normal shock and discretize the normal distribution of ϵ. Then we recursively apply the discretized version of the iid normal shock to transition matrix associated with the random walk shock.

3. There is a famous trick, which is valid for a large class of utility functions, including CRRA functional form. Look for Deaton (1991) or Papers by Chris Carroll.