Note on Neoclassical Growth Model: Value Function Iteration + Chebyshev Regression

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1 Introduction

The algorithm to solve the neoclassical growth model below, using value function iteration and Chebyshev regression, is shown. If you are not familiar with Chebyshev regression, please see the note on that topic before proceed.

Problem 1 (Neoclassical Growth Model: Recursive Formulation)

$$V(K) = \max_{C,K'} \{u(C) + \beta V(K')\}$$

subject to

$$C + K' = F(K, 1) + (1 - \delta)K$$
$$C \geq 0$$
$$K' \geq 0$$

2 Overview

As we are going to do value function iteration, we approximate the value function using Chebyshev polynomials of order n. If we use $m = n + 1$ collocation points, we call it Chebyshev collocation. If we use $m > n + 1$ points, it's called Chebyshev regression. But there is no significant difference regarding the algorithm.

3 The Algorithm

Algorithm 1 (NGM: Value Function Iteration and Chebyshev Regression)

1. *Set the order of polynomials used for approximation n.*

2. *Set the number of collocation points used m. It should be the case that $m \geq n+1$. If $m = n+1$ it is called Chebyshev collocation method, and if $m > n + 1$, the method is called Chebyshev regression.*

3. *Set a tolerance parameter ϵ, which is used to judge whether the value function iteration converged or not.*

4. *Set upperbound and lowerbound of the state space, $\underline{K}$ and $\overline{K}$. As we are interested in the dynamics below the steady state capital stock level, let's set $\overline{K}$ which is slightly higher than*

the steady state level (which can be computed analytically). As for $\underline{K}$, we can use a value slightly higher than zero. Including zero capital stock might be a problem as it implies zero consumption, and computing utility associated with zero consumption is critical, since the computer usually refuse to compute log of zero.

5. *Compute the collocation points $\{x_i\}_{i=1}^m$ implied by Chebyshev collocation points (or extended Chebyshev array) $\{z_i\}_{i=1}^m$. Notice that we do not have a control over how the collocation points are allocated over $[\underline{K}, \overline{K}]$.*

6. *We approximate the value function $V(K)$ by:*

$$\tilde{V}(K) = \sum_{j=0}^{n} \theta_j T_j(t(K))$$

   where $t(K)$ is a function which converts a value $K \in [\underline{K}, \overline{K}]$ to the corresponding value $z \in [-1, 1]$. Notice that the value function is characterized by coefficients $\{\theta_i\}_{j=0}^n$.

7. *Set an initial guess $\{\theta_i^0\}_{j=0}^n$. First choose a guess for the level of the value function at $\{x_i\}_{i=1}^m$. Call them $\{y_i^0\}_{i=1}^m$.*

8. *Using the Chebyshev regression method to obtain corresponding guess for the coefficients $\{\theta_i^0\}_{j=0}^n$.*

9. *Denote the guess of the value function implied by $\{\theta_i^0\}_{j=0}^n$ as $\tilde{V}^0(K)$.*

10. *For each $i = 1, 2, ..., m$, solve the following problem:*

$$g_i = \arg\max_{K' \in [\underline{K}, \overline{K}]} \{u(F(K_i, 1) + (1-\delta)K_i - K') + \beta \tilde{V}^0(K')\}$$

   It is necessary to use one of the one-dimension optimization algorithm to find an optimum g_i. Once g_i is obtained, use it to update value function. Specifically:

$$y_i^1 = u(F(K_i, 1) + (1-\delta)K_i - g_i) + \beta \tilde{V}^0(g_i)$$

11. *Using $\{y_i^1\}_{i=1}^m$ and Chebyshev collocation method, obtain updated guess for the coefficients $\{\theta_i^1\}_{j=0}^n$.*

12. *Compare $\{\theta_i^0\}_{j=0}^n$ and $\{\theta_i^1\}_{j=0}^n$. In particular, if:*

$$\max_i |\theta_i^0 - \theta_i^1| < \epsilon$$

   holds, we are done. Treat $\tilde{V}^1(K)$ as the optimal value function. $\{g_i\}_{i=1}^m$ are used to construct optimal decision rule function (you can use approximation method of your choice).

13. *Otherwise, update the value function by:*

$$\theta_i^0 = \theta_i^1 \qquad \forall i$$

   and go back to step 9.

14. *After we obtain convergence (according to our predetermined criteria), we should check the validity of the settings. First of all, check if the bounds on the state space are not binding. In particular, check if:*

$$\underline{K} < \min_K \overline{g}(K)$$

and

$$\overline{K} > \max_K \overline{g}(K)$$

hold.

15. *We also increase n, increase m, and reduce ϵ and make sure that the results are not sensitive to the changes.*